The Benedetto-Fickus Theorem

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May 7, 2025

A wild Riemannian Manifold appears

Given n lines in $\mathbb{C}^d$, represent each by a unit vector and form a matrix

$$Z = \begin{bmatrix} | & | & & | \\ z_1 & z_2 & \cdots & z_n \\ | & | & & | \end{bmatrix}$$

All such matrices form a Riemannian manifold $S(d, n) \subseteq \mathbb{C}^{d \times n}$ with $\langle X, Y \rangle_F = \mathrm{tr}(X^* Y)$

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A matrix Z is **tight** if it satisfies some generalization of the pythagorean theorem, or more simply if $ZZ^* = \frac{n}{d} I$ or the rows or orthogonal.

Finding Optimal Packings

Welch Bound

Define the **frame potential** as $FP : S(d, n) \to \mathbb{R}$ by

$$FP(Z) := \|Z^* Z\|_{\mathbb{F}}^2$$

When $n > d$ we have that

$$FP(Z) \geq \frac{n^2}{d}$$

With equality iff Z is tight

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Every matrix Z that is tight is a critical point of FP.

The Strong Benedetto-Fickus Theorem III

Theorem (Mixon, Needham, Shonkwiler, Villar)

Assume $n > d$ and let $F(Z, t)$ be the gradient flow on $S(d, n)$, where

- $F(Z, 0) = Z$
- $\frac{d}{dt} F(Z, t) = -\mathrm{grad} FP(F(Z, t))$

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$$\lim_{t\to\infty} F(Z_0, t)$$

is tight.

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This is telling us that the only local minimizers of FP are the matrices that are tight.